# An explicit duality for quasi-homogeneous ideals

Jean-Pierre Jouanolou

Université Louis Pasteur,

7 rue René Descartes,

67084 Strasbourg Cedex, France.

Email: jouanolo@math.u-strasbg.fr

8th February 2008

**Abstract**

Given $r \geq n$ quasi-homogeneous polynomials in $n$ variables, the existence of a certain duality is shown and explicited in terms of generalized Morley forms. This result, that can be seen as a generalization of [3, corollary 3.6.1.4] (where this duality is proved in the case $r = n$), was observed by the author at the same time. We will actually closely follow the proof of (loc. cit.) in this paper.

## 1 Notations

Let $k$ be a non-zero unitary commutative ring. Suppose given an integer $n \geq 1$, a sequence $(m_1, \ldots, m_n)$ of positive integers and consider the polynomial $k$-algebra $C := k[X_1, \ldots, X_n]$ which is graded by setting

$$\deg(X_i) := m_i \text{ for all } i \in \{1, \ldots, n\} \text{ and } \deg(u) = 0 \text{ for all } u \in k. \tag{1}$$

We will suppose moreover given an integer $r \geq n$, a sequence $(d_1, \ldots, d_r)$ of positive integers and, for all $i \in \{1, \ldots, r\}$, a (quasi-)homogeneous polynomial of degree $d_i$

$$f_i(X_1, \ldots, X_n) := \sum_{\substack{\alpha_1, \ldots, \alpha_n \geq 0 \\ \sum_{i=1}^n \alpha_i m_i = d_i}} u_{i,\alpha} X_1^{\alpha_1} \ldots X_n^{\alpha_n} \in k[X_1, \ldots, X_n]_{d_i}.$$

In the sequel, we denote by $I$ the ideal of $C$ generated by the polynomials $f_1, \ldots, f_r$, by $\mathfrak{m}$ the ideal of $C$ generated by the variables $X_1, \ldots, X_n$ and by $B$ the quotient $C/I$. We also set $\delta := \sum_{i=1}^r d_i - \sum_{i=1}^n m_i$.

## 2 The transgression map

Consider the Koszul complex $K^\bullet(f_1, \ldots, f_r; C)$, which is a $\mathbb{Z}$-graded complex of $C$-modules, associated to the sequence $(f_1, \ldots, f_r)$ of elements in $C$. It is of the

form

$$
\begin{tikzcd}
0 \arrow[r] & C(-\sum_{i=1}^r d_i) \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & \oplus_{i=1}^r C(-d_i) \arrow[r, "(f_1,\ldots,f_r)"] \arrow[d, equal] & C \arrow[r] \arrow[d, equal] & 0 \\
& K^{-r} & & K^{-1} & K^0 &
\end{tikzcd}
$$

where, more precisely, $K^{-i} := \bigwedge^i(K^{-1}) = \bigoplus_{J\subset\{1,\ldots,r\},|J|=i} C(-\sum_{j\in J} d_j)$. It gives rise to two classical spectral sequences

$$
\left\{\begin{array}{lllll}
{}'_1E^{pq} & := & H^q_{\mathfrak{m}}(K^p) & \Rightarrow & H^{p+q}_{\mathfrak{m}}(K^\bullet) \\
{}''_2E^{pq} & := & H^p_{\mathfrak{m}}(H^q(K^\bullet)) & \Rightarrow & H^{p+q}_{\mathfrak{m}}(K^\bullet).
\end{array}\right.
$$

Since $H^i_{\mathfrak{m}}(C) = 0$ if $i \neq n$, the first spectral sequence shows that, for all $p \in \mathbb{Z}$, $H^p_{\mathfrak{m}}(K^\bullet)$ is the cohomology module $H^{-n+p}(H^n_{\mathfrak{m}}(K^\bullet))$. Then, the second spectral sequence gives a transgression map, for all $p \in \{0, \ldots, r-n\}$,

$$
H^{-n-p}(H^n_{\mathfrak{m}}(K^\bullet)) \to H^0_{\mathfrak{m}}(H^{-p}(K^\bullet)). \tag{2}
$$

In particular, taking $p = r - n$ and using the equality

$$
H^{-r}(H^n_{\mathfrak{m}}(K^\bullet)) \simeq H^0(f_1, \ldots, f_r; H^n_{\mathfrak{m}}(C))(-\sum_{i=1}^r d_i),
$$

we get the transgression map

$$
\tau : H^0(f_1, \ldots, f_r; H^n_{\mathfrak{m}}(C))(-\sum_{i=1}^r d_i) \to H^0_{\mathfrak{m}}(H_{r-n}(K_\bullet))
$$

(note that we now use the more usual homological notation for the Koszul complex: $K_p = K^{-p}$ and $H_p(K_\bullet) = H^{-p}(K^\bullet)$ for all $p \in \mathbb{Z}$) which is particularly interesting because of the

**Proposition 1** *If* $\mathrm{depth}_I(C) = n$ *then* $\tau$ *is an isomorphism.*

*Proof.* If $\mathrm{depth}_I(C) = n$ then $H_i(K_\bullet) = 0$ for all $i > r - n$, and then the comparison of the two spectral sequences above shows immediately that $\tau$ is an isomorphism. $\square$

**Remark 2** *Observe that* $\mathrm{depth}_I(C) = n$ *if and only if* $\mathrm{depth}_I(C) \geq n$ *since* $I \subset \mathfrak{m}$ *and* $\mathrm{depth}_{\mathfrak{m}}(C) = n$.

We will denote, for all $\nu \in \mathbb{Z}$, by $\tau_\nu$ the homogeneous component of degree $\nu$ of $\tau$. Recall that, for all $\nu \in \mathbb{Z}$ we have a canonical perfect pairing between free $k$-modules of finite type

$$
C_\nu \otimes_k H^n_{\mathfrak{m}}(C)_{-\nu-\sum_{i=1}^r m_i} \to H^n_{\mathfrak{m}}(C)_{-\sum_{i=1}^r m_i} \simeq k. \tag{3}
$$

It follows that

$$
H^0(f_1, \ldots, f_r; H^n_{\mathfrak{m}}(C))(-\sum_{i=1}^r d_i)_\nu \simeq \mathrm{Hom}_k(B_{\delta-\nu}, k) =: \check{B}_{\delta-\nu}
$$

( $\check{}$ stands for the dual over $k$) which allows to identify $\tau_\nu$ with the $k$-modules morphism

$$\hat{\tau}_\nu : \check{B}_{\delta-\nu} \to H^0_{\mathfrak{m}}(H_{r-n}(K_\bullet))_\nu.$$

Observe that the direct sum $\bigoplus_{\nu\in\mathbb{Z}} \check{B}_{\delta-\nu}$ has a natural structure of $B$-module and so the $k$-linear map

$$\hat{\tau} := \bigoplus_{\nu\in\mathbb{Z}} \hat{\tau}_\nu : \bigoplus_{\nu\in\mathbb{Z}} \check{B}_{\delta-\nu} \to H^0_{\mathfrak{m}}(H_{r-n}(K_\bullet))$$

is a morphism of graded $B$-modules. It is clear that $\hat{\tau}$ is an isomorphism if $\tau$ is itself an isomorphism, and for instance if $\mathrm{depth}_I(C) \geq n$ by proposition 1. In the rest of this note we will give an explicit description of the map $\hat{\tau}$ in this case.

# 3 Generalized Morley forms

Introducing new indeterminates $Y_1, \ldots, Y_n$, we identify the ring $C \otimes_k C$ with the polynomial ring $k[\underline{X}, \underline{Y}]$ (we shortcut sequences: for instance $\underline{X}$ stands for the sequence $(X_1, \ldots, X_n)$) which is canonically graded via the tensor product: $\deg(X_i) = \deg(Y_i) = m_i$ for all $i = 1, \ldots, r$.

In $C \otimes_k C$, for all $i \in \{1, \ldots, r\}$ we choose a decomposition

$$\begin{aligned} f_i(X_1, \ldots, X_n) - f_i(Y_1, \ldots, Y_n) &= f_i \otimes_k 1 - 1 \otimes_k f_i \\ &= \sum_{j=1}^{n} (X_j - Y_j) g_{i,j}(X_1, \ldots, X_n, Y_1, \ldots, Y_n). \end{aligned} \tag{4}$$

Let $e_1, \ldots, e_r$ be the canonical basis of $\oplus_{i=1}^r C \otimes_k C(-d_i)$ with $\deg(e_i) = d_i$ for all $i \in \{1, \ldots, r\}$ and consider

$$\Delta := \sum_{\substack{\sigma\in\mathfrak{S}_r \text{ such that} \\ \sigma(1)<\cdots<\sigma(n), \\ \sigma(n+1)<\cdots<\sigma(r)}} \epsilon(\sigma) \begin{vmatrix} g_{\sigma(1),1} & g_{\sigma(1),2} & \cdots & g_{\sigma(1),n} \\ g_{\sigma(2),1} & g_{\sigma(2),2} & \cdots & g_{\sigma(2),n} \\ \vdots & \vdots & & \vdots \\ g_{\sigma(n),1} & g_{\sigma(n),2} & \cdots & g_{\sigma(n),n} \end{vmatrix} e_{\sigma(n+1)} \wedge \cdots \wedge e_{\sigma(r)}$$

where $\mathfrak{S}_r$ denotes the set of all the permutations of $r$ elements and $\epsilon(\sigma)$ the signature of such a permutation $\sigma \in \mathfrak{S}_r$.

**Lemma 3 ([1, 2.14.2])** *The element $\Delta$ is a cycle of the Koszul complex associated to the sequence $(f_1 \otimes_k 1 - 1 \otimes_k f_1, \ldots, f_r \otimes_k 1 - 1 \otimes_k f_r)$ in $C \otimes_k C$.*

We denote by $\Delta^\flat$ the class of $\Delta$ in the homology group

$$H_{r-n}(f_1(\underline{X}) - f_1(\underline{Y}), \ldots, f_r(\underline{X}) - f_r(\underline{Y}); C \otimes_k C)_\delta.$$

Note that, as a consequence of the so-called Wiebe lemma (see for instance [2, 3.8.1.7]), $\Delta^\flat$ does not depend on the choice of the decompositions (4) since the sequence $(X_1 - Y_1, \ldots, X_n - Y_n)$ is regular in $C \otimes_k C$.

**Lemma 4** *For all $P \in C$ we have $(P \otimes_k 1 - 1 \otimes_k P)\Delta^\flat = 0$, or in other words $P(\underline{X})\Delta^\flat = P(\underline{Y})\Delta^\flat$.*

*Proof.* First, it is clear that for all $i = 1, \ldots, n$ we have $(X_i - Y_i)\Delta^\flat = 0$. Indeed, each determinant fitting in the definition of $\Delta$ becomes an element of the ideal generated by the polynomials $f_j(\underline{X}) - f_j(\underline{Y})$, $j = 1, \ldots, r$, after multiplication by $(X_i - Y_i)$. The proof then follows from the equality

$$P(\underline{X}) - P(\underline{Y}) = \sum_{i=1}^{n} P(Y_1, \ldots, Y_{i-1}, X_i, \ldots, X_n) - P(Y_1, \ldots, Y_i, X_{i+1}, \ldots, X_n)$$

where each term in the above sum is divisible by at least one of the elements $(X_i - Y_i)$, $i \in \{1, \ldots, n\}$. $\square$

The canonical projection $C \otimes_k C \to C \otimes_k B$ induces a map

$$\begin{array}{c} H_{r-n}(f_1 \otimes_k 1 - 1 \otimes_k f_1, \ldots, f_r \otimes_k 1 - 1 \otimes_k f_r; C \otimes_k C) \\ \downarrow \\ H_{r-n}(f_1 \otimes_k 1, \ldots, f_r \otimes_k 1; C \otimes_k B) \end{array} \tag{5}$$

(note that $1 \otimes_k f_i = 0$ in $C \otimes_k B$ for all $i = 1, \ldots, r$) which sends $\Delta^\flat$ to an element, that we will denote $\nabla$, of degree $\delta$ in $H_{r-n}(f_1 \otimes_k 1, \ldots, f_r \otimes_k 1; C \otimes_k B)$.

Observe that, for all $q \in \mathbb{Z}$ the $C$-module $H_{r-n}(f_1 \otimes_k 1, \ldots, f_r \otimes_k 1; C \otimes_k B_q)$ is $\mathbb{Z}$-graded via the grading of $C$, so we deduce that the $B \otimes_k B$-module $H_{r-n}(f_1 \otimes_k 1, \ldots, f_r \otimes_k 1; C \otimes_k B)$ is bi-graded; for all $p, q \in \mathbb{Z} \times \mathbb{Z}$ we have

$$H_{r-n}(f_1 \otimes_k 1, \ldots, f_r \otimes_k 1; C \otimes_k B)_{p,q} := H_{r-n}(f_1 \otimes_k 1, \ldots, f_r \otimes_k 1; C \otimes_k B_q)_p.$$

We can thus decompose $\nabla$ with respect to this bi-graduation and we define $\nabla = \sum_{(p,q) \in \mathbb{Z}^2} \nabla_{p,q}$ with

$$\nabla_{p,q} \in H_{r-n}(f_1 \otimes_k 1, \ldots, f_r \otimes_k 1; C \otimes_k B_q)_p.$$

**Lemma 5** *For all couple $(p, q) \in \mathbb{Z}^2$ we have*

$$\nabla_{p,q} \in H^0_{\mathfrak{m} \otimes_k B + B \otimes_k \mathfrak{m}}(H_{r-n}(f_1 \otimes_k 1, \ldots, f_r \otimes_k 1; C \otimes_k B))_{p,q}.$$

*Proof.* This lemma follows from lemma 4; for all $j \in \{1, \ldots, n\}$ we have the equality $(X_j \otimes_k 1 - 1 \otimes_k X_j)\nabla = 0$ which gives, by looking at the homogeneous components,

$$(X_j \otimes_k 1)\nabla_{p,q} = (1 \otimes_k X_j)\nabla_{p+1,q-1}$$

for all $(p, q) \in \mathbb{Z}^2$ such that $p + q = \delta$. By successive iterations we obtain

$$(X_j \otimes_k 1)^{q+1}\nabla_{p,q} = (1 \otimes_k X_j)^{q+1}\nabla_{p+q-1,-1} = 0$$

which shows that $(\mathfrak{m} \otimes_k B)^{nq+1}\nabla_{p,q} = 0$. Exactly in the same way we obtain $(B \otimes_k \mathfrak{m})^{np+1}\nabla_{p,q} = 0$. $\square$

Finally, let us emphasize that $\nabla_{\delta,0}$ has a simple description. For all $i \in \{1, \ldots, r\}$ we choose a decomposition

$$f_i(X_1, \ldots, X_n) = \sum_{j=1}^{n} X_j f_{i,j}(X_1, \ldots, X_n) \in C \tag{6}$$

and similarly to what we did above, we consider

$$\Lambda := \sum_{\substack{\sigma\in\mathfrak{S}_r \text{ such that} \\ \sigma(1)<\cdots<\sigma(n), \\ \sigma(n+1)<\cdots<\sigma(r)}} \epsilon(\sigma) \begin{vmatrix} f_{\sigma(1),1} & f_{\sigma(1),2} & \cdots & f_{\sigma(1),n} \\ f_{\sigma(2),1} & f_{\sigma(2),2} & \cdots & f_{\sigma(2),n} \\ \vdots & \vdots & & \vdots \\ f_{\sigma(n),1} & f_{\sigma(n),2} & \cdots & f_{\sigma(n),n} \end{vmatrix} e_{\sigma(n+1)} \wedge \cdots \wedge e_{\sigma(r)}.$$

It is, as $\Delta$, a cycle of the Koszul complex $K_\bullet(f_1,\ldots,f_r;C)$. We denote $\bar{\Lambda}$ its class in $H_{r-n}(K_\bullet(f_1,\ldots,f_r;C))_\delta$, class which is independent, by the Wiebe lemma, of the choice of the decompositions (6) since the sequence $(X_1,\ldots,X_n)$ is regular in $C$.

**Lemma 6** *We have $\nabla_{\delta,0} = \bar{\Lambda}$ in $H_{r-n}(K_\bullet(f_1,\ldots,f_r;C))_\delta$.*

*Proof.* Indeed, $\nabla_{\delta,0}$ is the image of $\nabla^\flat$ via the map

$$C \otimes_k C = k[\underline{X},\underline{Y}] \to C = k[\underline{X}] : P(\underline{X},\underline{Y}) \mapsto P(\underline{X},0)$$

and this shows immediately the claimed equality. □

# 4 The explicit duality

Suppose given $\nu \in \mathbb{Z}$ and $u \in \check{B}_{\delta-\nu} = \mathrm{Hom}_k(B_{\delta-\nu},k)$. The canonical morphism $\mathrm{id}_C \otimes_k u : C \otimes_k B_{\delta-\nu} \to C \otimes_k k \simeq C$ induces a map

$$H_{r-n}(f_1,\ldots,f_r;C\otimes_k B_{\delta-\nu})_\nu \xrightarrow{H_{r-n}(f_1,\ldots,f_r;\mathrm{id}_C\otimes_k u)} H_{r-n}(f_1,\ldots,f_r;C)_\nu$$

which sends $\nabla_{\nu,\delta-\nu}$ to an element that we will denote $\nabla^{(u)}_{\nu,\delta-\nu}$. Therefore, to any $u \in \check{B}_{\delta-\nu}$ we can associate an element in $H_{r-n}(f_1,\ldots,f_r;C)_\nu$. Denoting $D_k^{\mathrm{gr}}(B)$ the graded $B$-module of graded morphisms from $B$ to $k$, that is to say

$$D_k^{\mathrm{gr}}(B) := \mathrm{Hom}_k^{\mathrm{gr}}(B,k) = \bigoplus_{\nu\in\mathbb{Z}} \mathrm{Hom}_k(B,k)_\nu = \bigoplus_{\nu\in\mathbb{Z}} \mathrm{Hom}_k(B_{-\nu},k) = \bigoplus_{\nu\in\mathbb{Z}} \check{B}_{-\nu}$$

we obtain a map

$$\omega : D_k^{\mathrm{gr}}(B)(-\delta) \quad \to \quad H_{r-n}(f_1,\ldots,f_r;C) \tag{7}$$

and we have the

**Proposition 7** *The map $\omega$ is a graded morphism (i.e. of degree 0) of graded $B$-modules whose image is contained in $H^0_{\mathfrak{m}}(H_{r-n}(f_1,\ldots,f_r;C))$.*

*Proof.* Let us choose a couple $(q,\nu) \in \mathbb{Z}^2$ and pick up $b \in B_q$ and $u \in D_k^{\mathrm{gr}}(B)(-\delta)_\nu = \check{B}_{\delta-\nu}$. To prove the $B$-linearity of $\omega$ we have to prove that $\omega(bu) = b\omega(u)$.

On the one hand, $bu \in \check{B}_{\delta-\nu-q}$ so $\omega(bu) \in H_{r-n}(\underline{f};C)_{\nu+q}$ is, by definition, the image of $\nabla_{\nu+q,\delta-\nu-q}$ by the map

$$H_{r-n}(\underline{f};C\otimes_k B_{\delta-\nu-q})_{\nu+q} \to H_{r-n}(\underline{f};C)_{\nu+q}$$

induced by $C \otimes_k B_{\delta-\nu-q} \to C : c \otimes_k x \mapsto cu(bx)$, which is also the image of $(1 \otimes_k b)\nabla_{\nu+q,\delta-\nu-q} \in H_{r-n}(\underline{f}; C \otimes_k B_{\delta-\nu})_{\nu+q}$ by the map

$$H_{r-n}(\underline{f}; C \otimes_k B_{\delta-\nu})_{\nu+q} \to H_{r-n}(\underline{f}; C)_{\nu+q}$$

induced by $C \otimes_k B_{\delta-\nu} \to C : c \otimes_k y \mapsto cu(y)$.

On the other hand, $b\omega(u)$ is the image of $\nabla_{\nu,\delta-\nu}$ by the map

$$H_{r-n}(\underline{f}; C \otimes_k B_{\delta-\nu})_{\nu} \to H_{r-n}(\underline{f}; C)_{\nu}$$

induced by $C \otimes_k B_{\delta-\nu} \to C : c \otimes_k x \mapsto c_1 cu(x)$ where $c_1 \in C$ is such that $c_1 = b$ in $B = C/I$. It follows that $b\omega(u)$ is the image of $(b \otimes_k 1)\nabla_{\nu,\delta-\nu}$ by the map

$$H_{r-n}(\underline{f}; C \otimes_k B_{\delta-\nu})_{\nu+q} \to H_{r-n}(\underline{f}; C)_{\nu+q}$$

induced by $C \otimes_k B_{\delta-\nu} \to C : c \otimes_k x \mapsto cu(x)$.

Now, by lemma 4, we know that $(1 \otimes_k b - b \otimes_k 1)\nabla = 0$ which implies, looking at the homogeneous component of degree $(\nu+q, \delta-\nu)$, that

$$b\omega(u) = (b \otimes_k 1)\nabla_{\nu,\delta-\nu} = (1 \otimes_k b)\nabla_{\nu+q,\delta-\nu-q} = \omega(bu).$$

Finally, we have $D_k^{\mathrm{gr}}(B)(-\delta)_\nu = 0$ for all $\nu > \delta$ so the $B$-linearity of $\omega$ implies that $B_q \mathrm{Im}(\omega) = 0$ for all sufficiently large integer $q$, which is equivalent to $\mathfrak{m}^p \mathrm{Im}(\omega) = 0$ in $H_{r-n}(\underline{f}; C)$ for all sufficiently large integer $p$. $\square$

According to the above proposition 7, and abusing notation, from now on we will assume that $\omega$ denotes the map (7) co-restricted to $H^0_{\mathfrak{m}}(H_{r-n}(f_1, \ldots, f_r; C))$. We also define $\omega_\nu$ as the graded component of degree $\nu$ of $\omega$:

$$\begin{array}{rcl} \omega_\nu : D_k^{\mathrm{gr}}(B)(-\delta)_\nu = \check{B}_{\delta-\nu} & \to & H^0_{\mathfrak{m}}(H_{r-n}(f_1, \ldots, f_r; C))_\nu \\ u & \mapsto & \nabla^{(u)}_{\nu,\delta-\nu}. \end{array}$$

We are now ready to state the main result of this note.

**Theorem 8** *If* $\mathrm{depth}_I(C) = n$ *then* $\hat{\tau} = \omega$.

*Proof.* We will prove that $\hat{\tau}_\nu = \omega_\nu$ for all $\nu \in \mathbb{Z}$. Recall that under the hypothesis $\mathrm{depth}_I(C) \geq n$ the map $\tau$ , and hence $\hat{\tau}$, become an isomorphism.

First, since $H_{r-n}(f_1, \ldots, f_r; C)$ is a sub-quotient of

$$\bigwedge^{r-n}\left(\bigoplus_{i=1}^{r} C(-d_i)\right) = \bigoplus_{1 \leq i_1 < i_2 < \cdots < i_{r-n} \leq r} C(-d_{i_1} - d_{i_2} - \cdots - d_{i_{r-n}}),$$

we deduce that $H_{r-n}(f_1, \ldots, f_r; C)_\nu = 0$ for all $v < 0$ (note that the extreme case is obtained when $r = n$). It follows that $\omega_\nu$ and $\hat{\tau}_\nu$ are both the zero map if $\nu < 0$.

If $\nu > \delta$ then $H^0_{\mathfrak{m}}(C)(-\sum_{i=1}^r d_i)_\nu = 0$. Since by hypothesis, $\hat{\tau}$ is an isomorphism we deduce that

$$H^0_{\mathfrak{m}}(H_{r-n}(f_1, \ldots, f_r; C))_\nu = 0$$

and hence that $\omega_\nu$ and $\hat{\tau}_\nu$ are again both the zero map.

We now prove that $\omega_\delta = \hat{\tau}_\delta$. By definition,

$$\omega_\delta : \check{B}_0 \simeq k \to H^0_{\mathfrak{m}}(H_{r-n}(\underline{f};C))_\delta$$

sends any $\lambda \in k$ to $\lambda\nabla_{\delta,0} = \lambda\bar{\Lambda}$ (see lemma 6), so it is completely determined by the formula $\omega_\delta(1) = \bar{\Lambda}$. To explicit the map $\hat{\tau}_\nu$ we will use the functoriality property of $\tau$ (and hence of $\hat{\tau}$); in this order, we will specify the sequence $\underline{f} := (f_1, \ldots, f_r)$ or $\underline{X} := (X_1, \ldots, X_n)$ in $C$ under consideration with the obvious notation $\tau(\underline{f})$ or $\tau(\underline{X})$. The decomposition (6) gives a graded morphism of $C$-modules (recall $r \geq n$)

$$\oplus_{i=1}^r C(-d_i) \xrightarrow{M} \oplus_{i=1}^n C(-m_i)$$

which can be lifted to a graded morphism of complexes

$$K_\bullet(f_1, \ldots, f_r; C) \xrightarrow{\wedge^\bullet M} K_\bullet(X_1, \ldots, X_n; C).$$

Note that $\wedge^p(M)$ is the zero map for all $p > n$. Using the self-duality property of the Koszul complexes, we obtain by duality a graded morphism of graded complexes

$$K(X_1, \ldots, X_n; C)(-\delta) \to K(f_1, \ldots, f_r; C)$$

which is of the form, denoting $K_1 := \oplus_{i=1}^r C(-d_i)$,

$$\begin{array}{ccccccccc}
C(-\sum_{i=1}^r d_i) & \longrightarrow & \cdots & \xrightarrow{\underline{X}} & C(-\delta) & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & 0 \\
\downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle \text{“}\Lambda\text{“}} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\
C(-\sum_{i=1}^r d_i) & \longrightarrow & \cdots & \longrightarrow & \wedge^{r-n}K_1 & \longrightarrow & \wedge^{r-n-1}K_1 & \longrightarrow \cdots \xrightarrow{\underline{f}} & C
\end{array}$$

By functoriality of the transgression map $\tau$ for morphisms of complexes, we obtain the commutative diagram

$$\begin{array}{ccc}
H^0(X_1, \ldots, X_n; H^n_{\mathfrak{m}}(C))(-\sum_{i=1}^r d_i) & \xrightarrow{\tau(\underline{X})(-\delta)} & H^0_{\mathfrak{m}}(C/\mathfrak{m})(-\delta) = k(-\delta) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle 1\mapsto\Lambda} \\
H^0(f_1, \ldots, f_r; H^n_{\mathfrak{m}}(C))(-\sum_{i=1}^r d_i) & \xrightarrow{\tau(\underline{f})} & H^0_{\mathfrak{m}}(H_{r-n}(f_1, \ldots, f_r; C))
\end{array}$$

which yields in degree $\delta$ the commutative diagram

$$\begin{array}{ccc}
k = H^n_{\mathfrak{m}}(C)_{-\sum_{i=1}^n m_i} & \xrightarrow{\hat{\tau}_0(\underline{X})} & k \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle 1\mapsto\Lambda} \\
k = H^n_{\mathfrak{m}}(C)_{-\sum_{i=1}^n m_i} & \xrightarrow{\hat{\tau}_\delta(\underline{f})} & H^0_{\mathfrak{m}}(H_{r-n}(f_1, \ldots, f_r; C))_\delta
\end{array}$$

Since the map $\hat{\tau}_0(\underline{X})$ is the identity [2, 2.6.4.6], we deduce that for all $\lambda \in k$ we have

$$\hat{\tau}_\delta(\underline{f})(\lambda) = \lambda\bar{\Lambda} \in H^0_{\mathfrak{m}}(H_{r-n}(f_1, \ldots, f_r; C))_\delta$$

and hence that $\hat{\tau}_\delta = \omega_\delta$.

Finally, assume that $0 \leq \nu < \delta$. By $B$-linearity of $\hat{\tau}$ and $\omega$ (see proposition 7), for all $u \in D_k^{\mathrm{gr}}(B)(-\delta)_\nu = \check{B}_{\delta-\nu}$ and for all $b \in B_{\delta-\nu}$ we have

$$b(\hat{\tau}_\nu(u) - \omega_\nu(u)) = \hat{\tau}_\delta(bu) - \omega_\delta(bu) = 0 \in H^0_{\mathfrak{m}}(H_{r-n}(f_1, \ldots, f_r; C))_\delta$$

with $\hat{\tau}_\nu(u) - \omega_\nu(u) \in H^0_{\mathfrak{m}}(H_{r-n}(f_1, \ldots, f_r; C))_\nu$. Since $H_{r-n}(\underline{f}; C)$ is a $B$-module, we have, for all $\nu \in \mathbb{Z}$ a canonical $k$-linear pairing

$$B_{\delta-\nu} \otimes_k H_{r-n}(\underline{f}; C)_\nu \to H_{r-n}(\underline{f}; C)_\delta.$$

By hypothesis, $\tau$ is an isomorphism and therefore we have the commutative diagram

$$\begin{tikzcd}
B_{\delta-\nu} \otimes_k H^0(\underline{f}; H^n_{\mathfrak{m}}(C))_{\nu-\sum_{i=1}^r d_i} \arrow[r] \arrow[d, "\wr"', "\mathrm{id}\otimes_k \tau_\nu"] & H^0(\underline{f}; H^n_{\mathfrak{m}}(C))_{-\sum_{i=1}^n m_i} \arrow[d, "\wr"', "\mathrm{id}\otimes_k \tau_\delta"] \\
B_{\delta-\nu} \otimes_k H^0_{\mathfrak{m}}(H_{r-n}(\underline{f}; C))_\nu \arrow[r] & H^0_{\mathfrak{m}}(H_{r-n}(\underline{f}; C))_\delta
\end{tikzcd}$$

where both vertical arrows are isomorphisms. Now, the top row being a non-degenerated pairing by (3), we deduce that the bottom row is also a non-degenerated pairing and hence that $\hat{\tau}_\nu = \omega_\nu$, as claimed. $\square$

**Corollary 9** *If* $\operatorname{depth}_I(C) = n$ *then* $\omega$ *is an isomorphism of* $B$*-modules.*


## Acknowledgment

I thank heartly Laurent Busé who proposed me to write down my notes and to translate them into english. He did a good job, but refused to be a co-author of the paper.


## References


[1] J. P. Jouanolou. Idéaux résultants. *Adv. in Math.*, 37(3):212–238, 1980.

[2] J. P. Jouanolou. Aspects invariants de l'élimination. *Adv. Math.*, 114(1):1–174, 1995.

[3] Jean-Pierre Jouanolou. Résultant anisotrope, compléments et applications. *Electron. J. Combin.*, 3(2):Research Paper 2, approx. 91 pp. (electronic), 1996. The Foata Festschrift.